Phys 401, Spring 2021

Homework #0, Due Friday, 29 January, 2021

These are physics and math skills that you will need for Phys 401. You do not need to derive any of these results. However, you should be able to utilize all of these math skills in a quiz or exam situation. Hence, please review any concepts that present difficulty. Complete the following by hand (no assistance from computers!):

1. The imaginary unit is $i = \sqrt{-1}$, and satisfies $i^2 = -1$. Use the Euler formula to expand $e^{i\theta}$ for real θ.
2. Given $z = x + iy$, where x and y are real, find expressions for z^* (the complex conjugate), $|z|$ (the magnitude of z), and the polar angle θ for this number.
3. Given a function $f(z) = a\,z + bz^2$ with $z = x + iy$, and x, y, a and b all real, find the imaginary part of the function $Im[f]$ in terms of only real quantities.
4. Given the three Cartesian unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$, calculate the following:
    a. $\hat{x} \times \hat{y}$
    b. $|\hat{x}|$
    c. $\hat{x} \cdot \hat{y}$
5. Given the vectors $\vec{r} = (r_x, r_y, r_z)$ and $\vec{s} = (s_x, s_y, s_z)$, calculate the cross product vector $\vec{r} \times \vec{s}$ in terms of its Cartesian components.
6. Find the eigenvalues and eigenvectors of this matrix: $\bar{\bar{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. In other words find all the λ and $\vec{v}$ that satisfy $\bar{\bar{A}}\vec{v} = \lambda\vec{v}$ [For a review of linear algebra, see Appendix A of Griffiths QM.]
7. What is the determinant of $\bar{\bar{A}}$ and how is it related to the eigenvalues?
8. What is the trace of $\bar{\bar{A}}$ and how is it related to the eigenvalues?
9. Prove that the matrix $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is *Hermitian*.
10. State two properties of Hermitian matrices.
11. What does it mean if two vectors are "linearly independent"? What does it mean to have a set of vectors that "span a space"?
12. Given a well-behaved scalar function of position $\chi(\vec{r})$ (e.g. the temperature distribution on the surface of the earth), what can we say is always true about the curl of the gradient of χ? (i.e. $\vec{\nabla} \times (\vec{\nabla}\chi)$)
13. Given a vector field $\vec{F} = k(x, 2y^2, 3z^3)$, where k is a constant, calculate its curl, $\nabla \times \vec{F}$ in Cartesian coordinates. What is the physical interpretation of this vector curl?
14. Calculate the vector divergence of $\vec{F}$, namely $\vec{\nabla} \cdot \vec{F}$, in Cartesian coordinates. What is the physical interpretation of this vector divergence?

15. What is the general solution $x(t)$ to the second-order linear differential equation $\ddot{x} = -\omega^2 x$, where ω is a real positive number and $\ddot{x} = d^2x/dt^2$?
16. What is the general solution $x(t)$ to the second-order linear differential equation $\ddot{x} = +k^2 x$, where k is a real positive number?
17. Given $ln(y) = b\ ln(x)$, where b is a constant, find y as a function of x, $y(x)$.
18. Evaluate $I = \int_{-2}^{3} 5x\, dx$.
19. Solve in closed form the indefinite integral $\int x\ \sin(x)\ dx$ by employing "integration by parts".
20. Expand $y(x) = \ln(1+x)$ to second order for $x \ll 1$.
21. Write the series expansion for $y(x) = \frac{1}{1-x}$ valid for $-1 < x < 1$.
22. Recall the Dirac delta function $\delta(x)$, which is defined through the expression $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$ for any arbitrary (but well-behaved) function $f(x)$. Describe in words the properties of the delta function.
23. The Fourier transform of a function $f(x)$ can be written as $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\ dx$. What is the Fourier transform of the Dirac delta function $\delta(x)$? Interpret the result.
24. Given that $\int_{-\infty}^{\infty} e^{-ax^2}\ dx = \sqrt{\frac{\pi}{a}}$, solve for the integral $\int_{-\infty}^{\infty} e^{-ax^2}e^{-ikx}\ dx$ by the "completing the square" in the exponent trick. (Answer: $\sqrt{\frac{\pi}{a}}\ e^{-k^2/4a}$)
25. Using the figure below, express the spherical unit vectors $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ in terms of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ for a general point in space labeled by (r, θ, ϕ).
26. Write down the differential volume element d^3r in spherical coordinates. Use the figure below for definition of the spherical coordinates.